A note on $N = 4$ supersymmetric mechanics on Kähler manifolds.

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The geometric models of $N = 4$ supersymmetric mechanics with $(2d.2d)_{\mathbb{C}}$-dimensional phase space are proposed, which can be viewed as one-dimensional counterparts of two-dimensional $N = 2$ supersymmetric sigma-models by Alvarez-Gaumé and Freedman. The related construction of supersymmetric mechanics whose phase space is a Kähler supermanifold is considered. Also, its relation with antisymplectic geometry is discussed.

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I. INTRODUCTION

Supersymmetric mechanics attracts permanent interest since its introduction [1]. However, studies focussed mainly on the $N = 2$ case, and the most important case of $N = 4$ mechanics did not receive enough attention, though some interesting observations were made about this subject: let us mention that the most general $N = 4, D = 1, 3$ supersymmetric mechanics described by real superfield actions were studied in Refs. [2,3] respectively, and those in arbitrary D in Ref. [4]; in [5] $N = 4, D = 2$ supersymmetric mechanics described by chiral superfield actions were considered; the general study of supersymmetric mechanics with arbitrary N was performed recently in Ref. [6]. In the Hamiltonian language classical supersymmetric mechanics can be formulated in terms of superspace equipped with some supersymplectic structure (and corresponding non-degenerate Poisson brackets). After quantization the odd coordinates are replaced by the generators of Clifford algebra. It is easy to verify that the minimal dimension of phase superspace, which allows to describe a $D-$dimensional supersymmetric mechanics with *nonzero* potential terms, is $(2D.2D)$, while supersymmetry specifies both the admissible sets of configuration spaces and potentials.

In the present work we propose the $N = 4$ supersymmetric one-dimensional sigma-models (with and without central charge) on Kähler manifold $(M_0, g_{a\bar b}dz^a d\bar z^{\bar b})$, with $(2d.2d)_{\mathbb{C}}$-dimensional phase space equipped with the symplectic structure

$$\begin{gathered}\Omega = \omega_0 - i\partial\bar\partial\mathbf{g} = \\ = d\pi_a \wedge dz^a + d\bar\pi_a \wedge d\bar z^a + \\ + R_{a\bar b c\bar d}\eta^a_i \bar\eta^b_i dz^a \wedge d\bar z^b + g_{a\bar b} D\eta^a_i \wedge D\bar\eta^b_i\end{gathered} \tag{1}$$

where

$$\mathbf{g} = i g_{a\bar b}\eta^a\sigma_0\bar\eta^b, \quad D\eta^a_i = d\eta^a_i + \Gamma^a_{bc}\eta^a_i dz^a, \quad i = 1, 2 \tag{2}$$

while Γ^a_{bc}, $R_{a\bar b c\bar d}$ are respectively the connection and curvature of the Kähler structure. The odd coordinates η^a_i belong to the external algebra $\Lambda(M_0)$, i.e. they transform as dz^a. This symplectic structure becomes canonical in the coordinates (p_a, χ^k)

$$\begin{gathered}p_a = \pi_a - \tfrac{i}{2}\partial_a\mathbf{g}, \quad \chi^m_i = \mathrm{e}^m_b\eta^b_i : \\ \Omega = dp_a \wedge dz^a + d\bar p_{\bar a} \wedge d\bar z^{\bar a} + d\chi^m_i \wedge d\bar\chi^{\bar m}_i,\end{gathered} \tag{3}$$

where e^m_a are the einbeins of the Kähler structure: $\mathrm{e}^m_a\delta_{m\bar m}\bar{\mathrm{e}}^{\bar m}_{\bar b} = g_{a\bar b}$. So, to quantize this model, one chooses

$$\hat p_a = -i\frac{\partial}{\partial z^a}, \quad \hat{\bar p}_{\bar a} = -i\frac{\partial}{\partial \bar z^{\bar a}}, \quad [\hat\chi^m_i, \hat{\bar\chi}^{\bar n}_j]_+ = \delta^{m\bar n}\delta_{ij}.$$

We restrict ourselves by the supersymmetric mechanics whose supercharges are *linear* in the Grassmann variables $\eta^a_i, \bar\eta^{\bar a}_i$. These systems can be obtained by dimensional reduction from $N = 2$ supersymmetric $(1+1)-$dimensional sigma-models by Alvarez-Gaumé and Freedman [8]; in the simplest case of $d = 1$ and in the absence of central charge these systems coincide with the $N = 4$ supersymmetric mechanics described by the chiral superfield action [5]. The constructed systems are connected also with the $N = 4$ supersymmetric mechanics describing the low-energy dynamics of monopoles and dyons in $N = 2, 4$ super-Yang-Mills theories [7].

We also propose the related construction of $N = 2$ supersymmetric mechanics whose phase superspace is the external algebra of an arbitrary Kähler manifold. Under the additional assumption that the base manifold is a hyper-Kähler one, this system should get the $N = 4$ supersymmetry. The relation of this system with antisymplectic geometry is discussed.

II. SIGMA-MODEL WITH STANDARD $N = 4$ SUSY.

Let us consider a one-dimensional supersymmetric sigma-model on an arbitrary Riemann manifold $(M_0, g_{\mu\nu}(x)dx^\mu dx^\nu)$, with $(2D.2D)-$dimensional phase superspace equipped with a supersymplectic structure

$$\begin{gathered}\Omega = d\left(p_\mu dx^\mu + \theta^\mu_i g_{\mu\nu}D\theta^\nu_i\right) = \\ = dp_\mu \wedge dx^\mu + \tfrac12 R_{\mu\nu\lambda\rho}\theta^\mu_i\theta^\nu_i dx^\lambda \wedge dx^\rho \\ + g_{\mu\nu}D\theta^\mu_i \wedge D\theta^\nu_i,\end{gathered} \tag{4}$$

where

$$D\theta^\mu_i = d\theta^\mu_i + \Gamma^\nu_{\rho\lambda}\theta^\rho_i dx^\lambda,$$

and $\Gamma^\mu_{\nu\lambda}$, $R_{\mu\nu\lambda\rho}$ are respectively the Cristoffel symbols and curvature tensor of underlying metric $g_{\mu\nu}dx^\mu dx^\nu$. On this phase superspace one can formulate the one-dimensional $N=2$ supersymmetric sigma-model with supercharges linear on Grassmann variables, viz

$$\begin{gathered}Q_1 = p_\mu\theta^\mu_1 + U_{,\mu}(x)\theta^\mu_2, \quad Q_1 = p_\mu\theta^\mu_2 - U_{,\mu}(x)\theta^\mu_1,\\ \mathcal{H} = \frac{1}{2}g^{\mu\nu}\left(p_\mu p_\nu + U_{,\mu}U_{,\nu}\right) + U_{\mu;\nu}\theta^\mu_1\theta^\nu_2 +\\ +R_{\mu\nu\lambda\rho}\theta^\mu_1\theta^\nu_2\theta^\lambda_1\theta^\rho_2 :\\ \{Q_i, Q_j\} = 2\delta_{ij}\mathcal{H}, \quad \{Q_i,\mathcal{H}\} = 0, i=1,2.\end{gathered} \tag{5}$$

One can also introduce a specific constant of motion ("fermionic number")

$$\mathcal{F} = g_{\mu\nu}\theta^\mu_1\theta^\nu_2 \;:\; \{\mathcal{F}, Q_i\} = \epsilon_{ij}Q_j \; \{\mathcal{F},\mathcal{H}\} = 0. \tag{6}$$

To get the $N=4$ supersymmetric one-dimensional sigma-model mechanics, one should require that the target space M_0 is a Kähler manifold $(M_0, g_{a\bar b}dz^a d\bar z^{\bar b})$, $g_{a\bar b} = \partial^2 K(z,\bar z)/\partial z^a\partial\bar z^b$ (this restriction follows also from the considerations of superfield actions: indeed, the $N-$extended supersymmetric mechanics obtained from the action depending on D real superfields, have a $(2D.ND)_{\mathbb{R}}$-dimensional symplectic manifold, whereas those obtained from the action depending on d chiral superfields have a $(2d.Nd/2)_{\mathbb{C}}-$dimensional phase space, with the configuration space being a $2d-$dimensional Kähler manifold). In that case the phase superspace can be equipped by the supersymplectic structure (1). The corresponding Poisson brackets are defined by the following non-zero relations (and their complex-conjugates):

$$\begin{gathered}\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \eta^b_i\} = -\Gamma^b_{ac}\eta^c_i,\\ \{\pi_a,\bar\pi_b\} = -R_{a\bar b c\bar d}\eta^c_k\bar\eta^d_k, \quad \{\eta^a_i, \bar\eta^b_j\} = g^{a\bar b}\delta_{ij}.\end{gathered}$$

To construct on this phase superspace the Hamiltonian mechanics with standard $N=4$ supersymmetry algebra

$$\begin{gathered}\{Q^+_i, Q^-_j\} = \delta_{ij}\mathcal{H},\\ \{Q^\pm_i, Q^\pm_j\} = \{Q^\pm_i,\mathcal{H}\} = 0, \quad i=1,2,\end{gathered} \tag{7}$$

let us choose the supercharges given by the functions

$$Q^+_1 = \pi_a\eta^a_1 + iU_{\bar a}\bar\eta^{\bar a}_2, \quad Q^+_2 = \pi_a\eta^a_2 - iU_{\bar a}\bar\eta^{\bar a}_1. \tag{8}$$

Then, calculating the commutators (Poisson brackets) of these functions, we get that the supercharges (8) belong to the superalgebra (7) when the functions $U_a, \bar U_{\bar a}$ are of the form

$$U_a(z) = \frac{\partial U(z)}{\partial z^a}, \quad \bar U_{\bar a}(\bar z) = \frac{\partial \bar U(\bar z)}{\partial\bar z^a}, \tag{9}$$

while the Hamiltonian reads

$$\begin{gathered}\mathcal{H} = g^{a\bar b}(\pi_a\bar\pi_b + U_a\bar U_{\bar b}) - iU_{a;b}\eta^a_1\eta^b_2 + i\bar U_{\bar a;\bar b}\bar\eta^{\bar a}_1\bar\eta^{\bar b}_2 -\\ -R_{a\bar b c\bar d}\eta^a_1\bar\eta^b_1\eta^a_2\bar\eta^d_2,\end{gathered} \tag{10}$$

where $U_{a;b} \equiv \partial_a\partial_b U - \Gamma^c_{ab}\partial_c U$.

The constant of motion counting the number of fermions, reads:

$$\mathcal{F} = ig_{a\bar b}\eta^a\sigma_3\bar\eta^{\bar b} : \quad \{Q^\pm_i, \mathcal{F}\} = \pm iQ^\pm_i, \; \{\mathcal{H},\mathcal{F}\} = 0. \tag{11}$$

Performing the Legendre transformation one gets the Lagrangian of the system

$$\begin{gathered}\mathcal{L} = g_{a\bar b}\dot z^a\dot{\bar z}^{\bar b} - \frac{1}{2}\eta^a_k g_{a\bar b}\frac{D\bar\eta^{\bar b}_k}{d\tau} + \frac{1}{2}\frac{D\eta^a_k}{d\tau}g_{a\bar b}\bar\eta^{\bar b} -\\ -g^{a\bar b}U_a\bar U_{\bar b} + iU_{a;b}\eta^a_1\eta^b_2 - i\bar U_{\bar a;\bar b}\bar\eta^{\bar a}_1\bar\eta^{\bar b}_2 +\\ +R_{a\bar b c\bar d}\eta^a_1\bar\eta^b_1\eta^c_2\bar\eta^d_2.\end{gathered} \tag{12}$$

The supersymmetry transformations of the Lagrangian are of the form

$$\begin{gathered}\delta^+_i z^a = \epsilon\eta^a_i,\\ \delta^+_i\eta^a_j = \epsilon\left(i\epsilon_{ij}\bar U_{\bar b}g^{\bar b a} + \Gamma^a_{bc}\eta^b_i\eta^c_j\right);\\ \delta^-_i z^a = 0,\\ \delta^-_i\eta^a_j = \epsilon\delta_{ij}\dot z^a\end{gathered} \tag{13}$$

where ϵ is an odd parameter: $p(\epsilon)=1$.
So, we get the action for one-dimensional sigma-model with four exact real supersymmetries. It can be straightly obtained by the dimensional reduction of $N=2$ supersymmetric $(1+1)$ dimensional sigma-model by Alvarez-Gaumé and Freedman [8] (the mechanical counterpart of this system without potential term was constructed in [10]). Notice that the above-presented $N=4$ SUSY mechanics for the simplest case, i.e. $d=1$, was obtained by Berezovoy and Pashnev [5] from the chiral superfield action

$$\mathcal{S} = \frac{1}{2}\int K(\Phi,\bar\Phi) + 2\int U(\Phi) + 2\int \bar U(\bar\Phi) \tag{14}$$

where Φ is chiral superfield. It seems to be obvious that a similar action depending on d chiral superfields will generate the above-presented $N=4$ SUSY mechanics.

III. $N=4$ SIGMA-MODEL WITH CENTRAL CHARGE

Let us consider a generalization of above system, which possesses $N=4$ supersymmetry with central charge

$$\begin{gathered}\{\Theta^+_i, \Theta^-_j\} = \delta_{ij}\mathcal{H} + \mathcal{Z}\sigma^3_{ij}, \quad \{\Theta^\pm_i, \Theta^\pm_j\} = 0,\\ \{\mathcal{Z},\mathcal{H}\} = \{\mathcal{Z},\Theta^\pm_k\} = 0.\end{gathered} \tag{15}$$

For this purpose one introduces the supercharges

$$\begin{gathered}\Theta^+_1 = \left(\pi_a + iG_{,a}(z,\bar z)\right)\eta^a_1 + i\bar U_{,\bar a}(\bar z)\bar\eta^{\bar a}_2,\\ \Theta^+_2 = \left(\pi_a - iG_{,a}(z,\bar z)\right)\eta^a_2 - i\bar U_{,\bar a}(\bar z)\bar\eta^{\bar a}_1,\end{gathered} \tag{16}$$

where the real function $G(z,\bar z)$ obeys the conditions

$$\partial_a\partial_b G + \Gamma^c_{ab}\partial_c G = 0, \quad G_{,a}(z,\bar z)g^{a\bar b}\partial_{\bar b}\bar U(\bar z) = 0. \tag{17}$$

The first equation in (17) is nothing but the Killing equation of the underlying Kähler structure (let us remind, that the isometries of the Kähler structure are Hamiltonian holomorphic vector fields) given by the vector

$$\mathbf{G} = G^a(z)\partial_a + \bar{G}^a(\bar{z})\bar{\partial}_a, \quad G^a = ig^{a\bar{b}}\bar{\partial}_b G. \tag{18}$$

The second equation means that the vector field $\mathbf{G}$ leaves the holomorphic function invariant:

$$\mathcal{L}_{\mathbf{G}} U = 0 \ \Rightarrow \ G^a(z)U_a(z) = 0.$$

Calculating the Poisson brackets of these supercharges, we get explicit expressions for the Hamiltonian

$$\begin{array}{c} \mathcal{H} \equiv g^{a\bar{b}}\left(\pi_a\bar{\pi}_{\bar{b}} + G_{,a}G_{\bar{b}} + U_{,a}\bar{U}_{,\bar{b}}\right) - \\ -iU_{a;b}\eta^a_1\eta^b_2 + i\bar{U}_{\bar{a};\bar{b}}\bar{\eta}^{\bar{a}}_1\bar{\eta}^{\bar{b}}_2 + \frac{1}{2}G_{a\bar{b}}(\eta^a_k\bar{\eta}^{\bar{b}}_k) - \\ -R_{a\bar{b}c\bar{d}}\eta^a_1\bar{\eta}^b_1\eta^c_2\bar{\eta}^d_2 \end{array} \tag{19}$$

and the central charge

$$\mathcal{Z} = i(G^a\pi_a + G^{\bar{a}}\bar{\pi}_{\bar{a}}) + \frac{1}{2}\partial_a\bar{\partial}_{\bar{b}}G(\eta^a\sigma_3\bar{\eta}^{\bar{b}}). \tag{20}$$

It can be checked by a straightforward calculation that the function $\mathcal{Z}$ indeed belongs to the center of the superalgebra (15). The scalar part of each phase with standard $N = 2$ supersymmetry can be interpreted as a particle moving on the Kähler manifold in the presence of an external magnetic field with strength $F = iG_{a\bar{b}}dz^a \wedge d\bar{z}^{\bar{b}}$ and in the potential field $U_{,a}(z)g^{a\bar{b}}\bar{U}_{,\bar{b}}(\bar{z})$.

The Lagrangian of the system is of the form

$$\begin{array}{c} \mathcal{L} = g_{a\bar{b}}\left(\dot{z}^a\dot{\bar{z}}^b + \frac{1}{2}\eta^a_k\frac{D\bar{\eta}^{\bar{b}}_k}{d\tau} + \frac{1}{2}\frac{D\eta^a_k}{d\tau}\bar{\eta}^{\bar{b}}\right) - \\ -g^{a\bar{b}}(G_aG_{\bar{b}} + U_a\bar{U}_{\bar{b}}) + \\ +iU_{a;b}\eta^a_1\eta^b_2 - i\bar{U}_{\bar{a};\bar{b}}\bar{\eta}^{\bar{a}}_1\bar{\eta}^{\bar{b}}_2 + R_{a\bar{b}c\bar{d}}\eta^a_1\bar{\eta}^b_1\eta^a_2\bar{\eta}^d_2. \end{array} \tag{21}$$

The supersymmetry transformations read

$$\begin{array}{c} \delta^+_i z^a = \epsilon\eta^a_i, \\ \delta^+_i \eta^a_j = \epsilon\left(i\epsilon_{ij}\bar{U}_{\bar{b}}g^{\bar{b}a} + \Gamma^a_{bc}\eta^b_i\eta^c_j\right), \\ \delta^-_i z^a = 0, \\ \delta^-_i \eta^a_j = \epsilon(\delta_{ij}\dot{z}^a - \epsilon_{ij}G^a). \end{array} \tag{22}$$

Assuming that $(M_0, g_{a\bar{b}}dz^a d\bar{z}^b)$ is the hyper-Kähler metric and that $U(z) + \bar{U}(\bar{z})$ is a tri-holomorphic function while the function $G(z,\bar{z})$ defines a tri-holomorphic Killing vector, one should get the $N = 8$ supersymmetric one-dimensional sigma-model. In that case instead of the phase with standard $N = 2$ SUSY arising in the Kähler case, we shall get the phase with standard $N = 4$ SUSY. The latter system can be viewed as a particular case of $N = 4$ SUSY mechanics describing the low-energy dynamics of monopoles and dyons in $N = 2, 4$ super-Yang-Mills theory [7]. Notice that, in contrast to the $N = 4$ mechanics suggested in the mentioned papers, in the above-proposed (hypothetic) construction also the four hidden supersymmetries could be explicitly written. We wish to consider this $N = 8$ supersymmetric mechanics, as well as its application to the solutions of super-Yang-Mills theory, in a forthcoming paper.

IV. THE RELATED CONSTRUCTION

Let us consider a supersymmetric mechanics whose phase superspace is the external algebra of the Kähler manifold $\Lambda(M)$, where $\left(M,\ g_{A\bar{B}}(z,\bar{z})dz^A d\bar{z}^{\bar{B}}\right)$ plays the role of the phase space of the underlying Hamiltonian mechanics. The phase superspace is a $(D.D)_{\mathbb{C}}-$ dimensional Kähler supermanifold equipped by the super-Kähler structure [11]

$$\begin{array}{c} \Omega = i\partial\bar{\partial}\left(K(z,\bar{z}) - ig_{A\bar{B}}\theta^A\bar{\theta}^{\bar{B}}\right) = \\ = i(g_{A\bar{B}} - iR_{A\bar{B}C\bar{D}}\theta^C\bar{\theta}^{\bar{D}})dz^A \wedge d\bar{z}^{\bar{B}} \\ +g_{A\bar{B}}D\theta^A \wedge D\bar{\theta}^{\bar{B}}, \end{array} \tag{23}$$

where $D\theta^A = d\theta^A + \Gamma^A_{BC}\theta^B dz^C$, and Γ^A_{BC}, $R_{A\bar{B}C\bar{D}}$ are respectively the Cristoffel symbols and curvature tensor of the underlying Kähler metrics $g_{A\bar{B}} = \partial_A\partial_{\bar{B}}K(z,\bar{z})$.

The corresponding Poisson bracket can be presented in the form

$$\{\ \ ,\ \ \} = i\tilde{g}^{A\bar{B}}\nabla_A \wedge \bar{\nabla}_{\bar{B}} + g^{A\bar{B}}\frac{\partial}{\partial\theta^A} \wedge \frac{\partial}{\partial\bar{\theta}^{\bar{B}}}\ , \tag{24}$$

where

$$\nabla_A = \frac{\partial}{\partial z^A} - \Gamma^C_{AB}\theta^B\frac{\partial}{\partial\theta^C}$$

and

$$\tilde{g}^{-1}_{A\bar{B}} = (g_{A\bar{B}} - iR_{A\bar{B}C\bar{D}}\theta^C\bar{\theta}^{\bar{D}}).$$

On this phase superspace one can immediately construct the mechanics with $N = 2$ supersymmetry

$$\{Q_+, Q_-\} = \tilde{\mathcal{H}}, \quad \{Q_\pm, Q_\pm\} = \{Q_\pm, \tilde{\mathcal{H}}\} = 0, \tag{25}$$

given by the supercharges

$$Q_+ = \partial_A H(z,\bar{z})\theta^A, \quad Q_- = \partial_{\bar{A}} H(z,\bar{z})\bar{\theta}^{\bar{A}} \tag{26}$$

where $H(z,\bar{z})$ is the Killing potential of the underlying Kähler structure,

$$\partial_A\partial_B H - \Gamma^C_{AB}\partial_C H = 0, \quad V^A(z) = ig^{A\bar{B}}\partial_{\bar{B}}H(z,\bar{z}).$$

The Hamiltonian of the system reads

$$\begin{array}{c} \tilde{\mathcal{H}} = g_{A\bar{B}}V^A\bar{V}^{\bar{B}} + iV^A_{,C}g_{A\bar{B}}\bar{V}^{\bar{B}}_{,\bar{D}}\theta^C\bar{\theta}^{\bar{D}} - \\ -R_{A\bar{B}C\bar{D}}V^A_{,C}\bar{V}^{\bar{B}}_{;\bar{D}}\theta^A\theta^C\bar{\theta}^{\bar{B}}\bar{\theta}^{\bar{D}}, \end{array} \tag{27}$$

while the supersymmetry transformations are given by the vector fields $\delta^\pm \equiv \{Q^\pm,\ \}$,

$$\delta^- = -iV^A(z)\frac{\partial}{\partial\theta^A} - iV^A_{;C}\theta^C\mathcal{N}^D_A\nabla_D, \tag{28}$$

where

$$\left(\mathcal{N}^{-1}\right)^A_B \equiv \delta^A_B - iR^A_{BC\bar{D}}\theta^C\bar{\theta}^{\bar{D}}.$$

Requiring that M be a hyper-Kähler manifold, we can double the number of supercharges and get a $N = 4$ supersymmetric mechanics. In that case the Killing potential should generate a tri-holomorphic vector field.

The phase space of the system under consideration can be equipped, in addition to the Poisson bracket corresponding to (23), with the antibracket (odd Poisson bracket) associated with the odd Kähler structure $\Omega_1 = i\partial\bar{\partial}K_1$, where $K_1 = \mathrm{e}^{i\alpha}\partial_A K(z,\bar{z})\theta^A + \mathrm{e}^{-i\alpha}\partial_{\bar{A}}K(z,\bar{z})\bar{\theta}^{\bar{A}}$, $\alpha = 0, \pi/2$,

$$\{\quad,\quad\}_1 = \mathrm{e}^{-i\alpha} g^{\bar{A}B}\nabla_{\bar{A}} \wedge \frac{\partial}{\partial\theta^B} + c.c. \quad . \tag{29}$$

It is easy to observe that the following equality holds [11]

$$\mathbf{L} \equiv \{\tilde{\mathcal{Z}},\ \} = \{Q,\ \}_1, \tag{30}$$

where

$$\begin{array}{c}\tilde{\mathcal{Z}} \equiv H(z,\bar{z}) + i\partial_A\partial_{\bar{B}}H(z,\bar{z})\theta^A\bar{\theta}^{\bar{B}} \\ Q = \mathrm{e}^{i\alpha}Q_+ + \mathrm{e}^{-i\alpha}Q_-.\end{array} \tag{31}$$

Then, after obvious algebraic manipulation with the Jacobi identities, one gets the following relations:

$$\{\tilde{\mathcal{Z}}, \tilde{\mathcal{H}}\} = \{\tilde{\mathcal{Z}}, Q_\pm\} = 0. \tag{32}$$

Hence, the function $\tilde{\mathcal{Z}}$ is a constant of motion, which belongs to the center of the superalgebra defined by $Q_\pm, \mathcal{H}, \tilde{\mathcal{Z}}$. One can also introduce another constant of motion, i.e. the "fermionic number"

$$\tilde{\mathcal{F}} = ig_{AB}\theta^A\bar{\theta}^{\bar{B}} : \ \{Q_\pm, \tilde{\mathcal{F}}\} = \pm iQ_\pm,\ \{\tilde{\mathcal{H}}, \tilde{\mathcal{F}}\} = 0. \tag{33}$$

Notice that the supermanifolds provided by the even and odd symplectic (and Kähler) structures, and particularly the equation (30), were studied in the context of the problem of the description of supersymmetric mechanics in terms of antibrackets [11,12] (this problem was suggested in [13]). Later this structure was found to be useful in equivariant cohomology, e.g. for the construction of equivariant characteristic classes and derivation of localization formulae [14], since the vector field (30) can be identified with the Lie derivative along the vector field generated by H, while the vector fields $\{\tilde{\mathcal{F}},\ \}_1$, $\{H,\ \}_1$ corresponds to external differential and operator of inner product. Hence, $\{H + \tilde{\mathcal{F}}, H + \tilde{\mathcal{F}}\}_1 = 2Q$, and $H + \tilde{\mathcal{F}}$ defines an equivariant Chern class; the Lie derivative of the even symplectic structure (23) along the vector field $\{H + \tilde{\mathcal{F}},\ \}_1$ yields the equivariant even pre-symplectic structure, generating equivariant Euler classes of the underlying Kähler manifold.

Notice also, that the antibrackets are the basic object of the Batalin-Vilkovisky formalism [15], while the pair of antibrackets, corresponding to the $\alpha = 0, \pi/2$, together with corresponding nilpotent vector fields $\{\mathcal{F},\ \}_1$ and the associated $\Delta-$operators, form the "triplectic algebra" underlying the BRST-antiBRST-invariant extension of Batalin-Vilkovisky formalism [16].

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